Measure Theory of Premeasures and Measures with Extension

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ABSTRACT

Measure theory is one of the important scenarios in mathematics as it has better properties as compared to set theory. We aimed to focus on the premeasures and measures construction along with their extension. Initially the Lebesgue measure was used at undergraduate mathematical study but recently there are many measures are available such as Borel, Gauss Measure, Euler Measure, Hausdorff Measure, Haar Measure and many more.In construction of few measures the Carath´eodory extension theorem as well as Carath´eodory-Hahn theorem was solicited subsequently. Similarly the Dynkin's $\pi - \lambda$ theorem was also used in construction purpose. This paper illustrated the mathematical analysis for construction of measures and premeasures. Also extensions of measures and premeasures were studied in this paper.

Keywords

σ-algebra, Premeasure, Measure, Extensions of measure and premeasure, Carath´eodory theorem.

Introduction

Measure theory deals with the branch of mathematics. It plays an important role in area of mathematical analysis where it is familiar with the basic notion and various statements. It does not much important to have detailed analysis proof if used. Consider the example, anyone should has a particular place where he or she could visited so at time it is not necessary to have any detailed proof. Measure theory is permissible or can be applying to the limiting processes but is not applicable in case of weaker setups such as Riemann integration theory. Basically measure is the way to assign volume to the subsets of R. The all set collection which allowed to measure must σ algebra. Very important thing is , the measure is used it for integrating the function [1].

Recent measure theories are remarkably very powerful as compared to previous theory as extra ordinary types of sets enough regular for measured, but still it has some limitations. Everybody is aware of length, area and volume concepts which are applicable specifically to reasonable regular set. Proper realization class of the sets allows its measure indispensable which further leads to existing non measurable sets. [2].

Measures Assigns the zero to empty set and then it be the additive. Measure of the large subset is breakdown or divided into finite number of small disjoint subsets .Additive aspect describes the sum of measures of small subset. Generally anyone wants to assign a predictable size to every subset for given set with satisfaction of the axioms of measure. Here travail examples follows this like measure counting (It defines measure of set S along with present no of elements of set S. Due to this scenario measurable subset are comes in picture which defines the measure on subset collection for all present subsets. Measurable subsets require forming the σ algebra. It means that finite unions, complements for measurable subsets and finite intersection are measurable [3].

1.1 Measures:

In field of mathematics, Lots of systematic ways are available for assigning the number for suitable subset which are made from that set, instinctively elucidated its size. In this sensation, measures are the generalization of hypothesis of area, length and volume [4].Fig 1 shows the measure i.e. formation of set by considering various parameter.

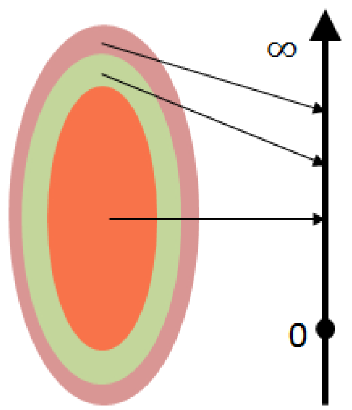

Fig 1: Measure

Term measures have specific technical definition which usually involves sigma algebra. It is strenuous to comprehend. However technical statement is very important as it is base for any mathematical analysis which includes some slippery sets of objectives of calculus. Let's consider example. Every definition or statements of integrals are based on particular measure. Riemann integral is depends on Jordan measure whereas Lebesgue integral is depends on the Lebesgue measure. Measure theory is nothing but the study of the measures with their application to the integration [5].

The non-negative real function from the delta ring D such that

$$s(\emptyset) = 0 \ \ where\ \emptyset\ represents\ empty\ set.$$

And

$m(B) = \sum \quad m(B_n)$ For Countable or finite collection of pairwise disjoint set B_n in D such that $B = \cup B_n$ is also present in D. If D is σ finite and m become bounded, then m is extending uniquely to measure defined on σ Algebra generated by D. The m is known as probability measure on set Z if m (Z) =1 and D is σ Algebra. Basically measure is real value function on power set P(S) of infinite set X. Probability measure satisfies the bellows property.

1. $\mu(\emptyset) = 0\ and\ \mu(S) = 1$
2. $If\ Z \subseteq Y\ then\ \mu(z) \leq\ \mu(y), \sigma$
3. $\mu(\{p\} = 0\ for\ all\ p\ \epsilon\ X$
4. $If\ S_n\ , n = 0,1,2,3 \dots \dots \dots \dots \dots \dots$ Are pairwise disjoint set, then

$$\mu(\cup_{n=0}^{\infty} Z_n) = \sum_{n=0}^{\infty} \quad \mu Z_n \dots \dots \dots \dots \dots \dots \dots \dots (1)$$

The measure m can extend by completion. A Subset of set with measure zero forms $\delta\ ring\ J$.From set J, the set D changes so $\delta\ ring\ D_c$ which make set D will completed with reference to obtained measure m.

The measure (m) is a completed if $D = D_c$.If measure is not completed then is extending to D_c by satisfying the condition $m\big((P \setminus Q) \cup (Q \setminus P)\big) = m(P)\ where\ P\epsilon D\ and\ Q\epsilon\ G$ [6].

The triple (P, Q, μ) of the non-empty set P. The σ algebra of subset of P and a measure μ on Q is termed measure space. Two conditions such as simplicity and no confusion, μ is measure on P or measure on Q [7] and [8].

1.2 Premeasures:

It is a function, in some part of sensation, precursor used for open, fair, regardless outcome result and honest measure for given space. The fundamental theorem is present in measure theory which states that premeasures can be extends to measure.

Let Z set and $A \subset P(Z)$ is algebra. Premeasure on algebra A is mapped $\rho: A \to [0, \infty]$ which follows satisfies

1. $\rho(\emptyset) = 0$
2. $If\ (P_i)_{i \in IN} A$ and $P = \bigcup_{i \in IN} P_i \in A$ then

$$\rho(P) = \sum_{i \in IN} \quad \rho(P_i)$$

Let's assume that R is the ring of subset (which is closed under relative complement and union) of fixes set say X. Here $\mu_0 : R$ and R varies from 0 to ∞ is set function. μ_0 termed as premeasure if it satisfies bellows conditions:

$$\mu_0(\emptyset) = 0 \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (2)$$

It represents the empty sets,

And for every finite sequence or countable sequence $\{P_n\}_{n \in N} \subseteq R$ of the pairwise disjoint set and its union lies in set R,

$$\mu_0(\cup_{n=1}^{\infty} P_n) = \sum_{n=1}^{\infty} \quad \mu_0(P_n) \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (3)$$

This property is known as σ additivity [7] and [8].

The addition or composition of two measurable functions with the continuous function must be again measurable. All the sensible settlements are correct but proven details or proof of that statement occupies large space.

I.3 Prerequisites mathematical Details

Definitions1.3.1: Let assume P is the set. A collection of m of subset of P is known as σ algebra in P, if it satisfies the following condition

1. $P \in m$
2. $P - A \in m\ for\ A \in m$

3.$\bigcup_{n=1}^{\infty} X_n \in m$ for finite collection where n varying from 1 to ∞ [9] and [10].

Definition1.3.2: s is the nonempty subset of set P is termed as semi ring where X and Y belongs to s then $(X \cap Y) \in s$. There are also presence of finite disjoint collection $\{K_l\}_{l=1}^{n}$ of set in s having condition $X - Y = \bigcup_{l=1}^{n} K_l$

Here collection of s of subsets of P is known as ring of set. It is closed with reference to evolution of finite unions and corresponding complements and hence with reference to evolution of finite or countable intersection [2] and [11].

Definition 1.3.3: From the measurable space, mean a pair (P, m) consist of set P and σ algebra m of subsets of P.A subset Z of P is termed as measurable with reference to m supplied Z ∈ m [12] and [13].

Definition1.3.4: From the measure space(P, Q, μ), means a measurable space (P, Q) together defines the measure μ on Q [13] and [14].

Definition 1.3.5: A countable additive unique se function s which is defined on delta ring or algebra A is known as premeasure on A and $s(\emptyset) = 0$ [12] and [13].

Definition 1.3.6: A set function $\mu^* = 2^z \to [0, \infty]$ is known as outer measure supplied $\mu^*(\emptyset) = 0$ and it is finitely single or unique [13].

I.4 Types of measure:

There are number of measure are present such as Radon Measure, Borel Measure, Gauss Measure, Euler Measure, Hausdorff Measure, Haar Measure, Integral, Helson-Szegö Measure, Lebesgue Measure, Jordan Measure, Mahler Measure, Liouville Measure, Minkowski Measure, Probability Measure, Wiener Measure and Ergodic Measure [7] and [8].

I.5 Properties of measure:

Proposition 1.5.1: Consider (P, Q, μ) is a measure space. P and Q belongs to A in such way

that $P \subseteq Q$ and hence $\mu(P) \subseteq \mu(Q)$.In case of addition, The subset Q satisfies $\mu(P) < \infty$

then $\mu(Q - P) \subseteq \mu(Q) - \mu(P)$.

Proof: The set P and $Q - P (= Q \cap P^c)$ are belongs to A and disjoints. Now then use the aspect countable or finite additivity of the μ.

Remark 1.5.2: Property of μ used in proposition 1.5.1 is known as monotonicity of μ.

Remark: 1.5.3 $\mu(P) < \infty$,this condition is extra which is given in proposition 1.5.1 because sometimes it is happens that $\mu(Q - P) < \infty$ but both $\mu(P) and\ \mu(Q)$ are equal to ∞.Let's see the example, $P = R$, $A = B(R)$ and μ is the length of measure on P. Consider Q= R,P=Set of irrational element in R.Then $Q - P = Q$ is the finite subset of R,hence $\mu(Q - P) = 0$.But $\mu(Q) = \mu(P) = \infty$ [15].

I.6 Construction of Premeasure:

Proposition1.6.1: Consider $s \subset 2^z$ is the semi algebra,$A = A(s) and\ \mu : A \to [0, \infty]$ is the finite additive measure. Hence μ is the premeasure on A if μ is the sub additive on s.

Proof: If μ is the premeasure on set A then μ becomes sigma (σ) additive. It becomes sub additive on set s.

Suppose $P = \sum_{n=1}^{\infty} P_n$ with $P \in A$ and each element $P_n \in A$ which can be written as

$$P = \sum_{j=1}^{k} E_j \ with\ E_j \in s \ldots\ldots\ldots\ldots\ldots (4)$$

And

$$P_n = \sum_{i=1}^{N_n} E_{n,i} \ with\ E_{,i} \in s \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Then get the equation

$$E_j = P \cap E_j = \sum_{n=1}^{\infty} P_n \cap E_j = \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} E_{n,i} \cap E_j \ldots\ldots\ldots\ldots\ldots. (6)$$

This equation is finite union and using assumption

$$\mu(E_j) \le \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \mu(E_{n,i} \cap E_j) \ldots\ldots\ldots\ldots\ldots\ldots.. (7)$$

Adding this equation on the j with using countable additivity property of μ shows

$$\mu(P) = \sum_{j=1}^{k} \mu(E_j)$$
$$\leq \sum_{j=1}^{k} \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \mu(E_{n,i} \cap E_j) \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(8)$$

$$= \sum_{j=1}^{k} \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \mu(E_{n,i} \cap E_j)$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \mu(E_{n,i})$$

$$= \sum_{n=1}^{\infty} \mu P_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(9)$$

Hence sub additivity of μ on the A is proved.

I.7 Construction of measure.

As μ is the premeasure on the A where $A \to [0, \infty]$ if $\mu(A_n) \uparrow \mu(P)$ for $\{P_n\}_{n=1}^{\infty} \subset A$ like $P_n \uparrow P \in A$ If the $\mu(P) < \infty$ then μ is the premeasure on algebra A if $\mu(A_n) \downarrow 0$ for $\{P_n\}_{n=1}^{\infty} \subset A$ since $P_n < \emptyset$.

Proposition1.7.1: Consider μ a premeasure on algebra A, the n the μ has unique extension to function A_σ , It satisfies the continuity, monotonicity, strong additivity, sub additivity and sigma additivity.

Proof: Assume that P, Q are the sets in the A_σ such way $P \subset Q$ .Consider $\{P_n\}_{n=1}^{\infty}$ *and* $\{P_n\}_{n=1}^{\infty}$ are queue in A as $P_n \uparrow P$ and $Q_n \uparrow Q$ where $n \to \infty$.Since $Q_m \cap P_n \uparrow P_n$ where $m \to \infty$.The continuity for measure μ on A.

$$\mu\big((P_n) = \mu(Q_m \cap P_n\big) \leq \mu\,(Q_m) \ldots\ldots\ldots\ldots\ldots(10)$$

Consider $n \to \infty$,

$$\mu(\,P_n) \leq \mu\,(Q_m) \ldots\ldots\ldots\ldots\ldots\ldots.(11)$$

With the help of above equation Q=P, suggested,

$$\mu(\,P_n) = \mu\,(Q_m) \ldots\ldots\ldots\ldots\ldots\ldots.(12)$$

Whenever

$P_n \uparrow P$ And $Q_n \uparrow Q$.

Hence it becomes unambiguous for defining $\mu(P)$ by

$$\mu(P) = \mu(P_n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

For some sequence $\{P_n\}_{n=1}^{\infty} \subset A$ such way $P_n \uparrow P$ with the help of definition, continuity of measure μ is understandable and monotonic of μ following the equation 11.

Consider $P, Q \epsilon A_\sigma$ and $\{P_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$ are the sequence in A such way $P_n \uparrow P$ and $Q_n \uparrow Q$ for $n \to \infty$ .It passes through identity

$$\mu(P_n \cup Q_n) + \mu(P_n \cap Q_n) = \mu(P_n) + \mu(Q_n) \ldots\ldots\ldots\ldots\ldots..(14)$$

Here μ is finitely additive on the A_σ.

Let consider $\{P_n\}_{n=1}^{\infty}$ is any sequence in the A_σ.select the $\{P_{n,i}\}_{i=1}^{\infty} \subset A$ which follows the $P_{n,i} \uparrow P_n$ since $i \to \infty$.

Then we get ,

$$\mu\big(\cup_n^N P_{n,N}\big) \leq \sum_{n=1}^{N} \mu\big(P_{n,N}\big) \leq \sum_{n=1}^{N} \mu(P_n)$$
$$\leq \sum_{n=1}^{\infty} \mu(P_n) \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

Here $A \ni \cup_{n=1}^{N} P_{n,N} \uparrow \cup_{n=1}^{N} P_n \epsilon A_\sigma$ .If $N \to \infty$, the previous equation holds the bellows equation of sub-additivity

$$\mu(\cup_{n=1}^{\infty} P_n) \leq \sum_{n=1}^{\infty} \mu(P_n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

Further assumption $\{P_{n,i}\}_{i=1}^{\infty} \subset A_\sigma$ is disjoints sequence with countable additivity along with monotonic property of μ on algebraA_σ. So finally get the equation

$$\sum_{n=1}^{\infty} \mu(P_n) = \sum_{n=1}^{N} \mu(P_n) = \mu(\cup_{n=1}^{N} P_n)$$
$$\leq \mu(\cup_{n=1}^{\infty} P_n) \ldots\ldots\ldots\ldots\ldots\ldots.(17)$$

Consider μ is the finite premeasure on algebra, $A \subset 2^Z$, and $P \in A_\delta \cap A_\sigma$.So we have two conditions $P, P^c \in A_\sigma$ and $Z = P \cup P^c$ which follows the $\mu(Z) = \mu(P) + \mu(P^c)$.

From all above observations μ is extended to the function on $A_\delta \cup A_\sigma$ with definition of

$\mu(P) \coloneqq \mu(Z) - \mu(P^c)$ for all values of $P \in A_\delta$ [16].

I.8 Outer measures from premeasure:

Theorem 1.8.1: Consider μ^* is the outer measure on 2^Z.Then collection m of sets are measurable with reference to μ^* is the sigma (σ) algebra. If μ is the limitations of μ^* to m, then (P,Q, μ) is entire measure space [7] and [13].

I.9 Extension of measure:

It is the easiest way to define measure on the basis of small collection of power set and stating the consistency condition which allows extending measure to power set. Power set is nothing but the probability measures on possible uncountable and infinite sets such as power set of naturals.

Theorem1.9.1: Carath´eodory's definition

For the outer measure$\mu^*: 2^Z \to [0,\infty]$, it is subset E of Z measurable supplied for each subset P of Z,

$\mu^*(P) = \mu^*(P \cap E) + \mu^*(P \cap E^c)$ [2],[14] and[15].

The outer measure m^* on Z is known as metric outer measure with respect to Carath´eodory if it satisfies,

$m^*(P_1 \cup P_2) = m^*(P_1) + m^*(P_2)$
Whenever $d(P_1, P_2)$ [17].

Theorem1.9.2: (Carath´eodory's Extension Theorem)

Consider μ is measure on algebra. μ* is outer measure introduced by μ.Then limitation $\underline{\mu}$ of μ* to μ* is measurable sets the extension of μ to σ algebra contains with A.It the μ is σ finite or countable, then $\underline{\mu}$ is only measure on small σ algebra contains with A which is the extension of μ [13] and [18].

1.9.3 Extension of premeasure: The Carath´eodory-Hahn Theorem

Let consider μ: s→ $[0,\infty]$ is premeasure on semiring s of subset of Z. Then Carath´eodory measure $\underline{\mu}$ introduces by μ is the extension of μ. Moreover, if μ is the σ finite =$\underline{\mu}$.The $\underline{\mu}$ is unique measure on σ algebra of μ* is measurable sets which extends μ [13].

Consider $\rho: A \to [0,\infty]$ is the premeasure on algebra A of Z. $M = \langle A \rangle$ is the algebra generating A.

1. It is special measure $\mu: M \to [0,\infty]$ such way that$\mu|M = \rho$.
2. If $v: M \to [0,\infty]$ such way $v|M = \rho$ then

$v(P) < \mu(P)$ For set $P \in M$ and $v(P) < \mu(P)\ if\ \mu(P) < \infty$.

Proof: From previous discussion it is clear that μ can be obtained by restriction of M.Now consider v is one more measure on M, such way $v|A = \mu$.There are few things about u and v.

$P_i \in A$ and $E = \cup_{i \in IN} P_i$ Then with approximation of $\mu(E)$ in the form of sum $\sum_{i \in IN} \mu(P_i)$,this is get for all elements of $E \in M$

$v(E) = \mu(E)$…………… (18)

Let $\mu(E) < \infty$ and $E \in M$,with approximation $P_i \in A$ on the basis of definition of ρ^0,will implies

$$\mu(P) = v(E)1 \leq$$

So finally get $v(E) = \mu(P)$ [19].

There are some other extension theorems also used such as Dynkin's π − λ theorem. In this there are two systems comes in picture i.e. π system and λ system. Collection of the subsets is closed in nature for the finite or countable intersections in case of π system while collection of subset is closed having disjoints and complementation finite unions in case of λ system. So the class having both π system and λ system becomes σ algebra [20].Carath´eodory-Hahn Theorem is important in case of product measure.

I.10 Conclusion:

From the above discussion, it is concluded that Measures have most formal properties than the sets. In this paper, we have studied the basics of measure and the premeasures along with their construction having mathematical analysis. Also focused the extensions for measures and premeasures using Carath´eodory theorem. Little bit basic part of Dynkin's π − λ theorem and Carath´eodory-Hahn Theorem is highlighted. This complete review paper introduces the measure theory.

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